Stabilization of coherent oscillations in spatially extended dynamical systems

P.-M. Binder and Juan F. Jaramillo
Departamento de Física, Universidad de los Andes, Apartado Aéreo 4976, Bogotá, Colombia

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We consider high-dimensional cellular automata with coherent period-three oscillations. We find that the interface dynamics between inhomogeneous phases has diffusivelike behavior. This phenomenon, not considered in a previous droplet-growth argument against the existence of coherent oscillations in spatially extended systems, explains why these models escape the argument. [S1063-651X(97)10108-8]

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A few years ago a generic argument was given by Bennett *et al.* against the existence of coherent oscillations with period larger than two in spatially extended systems with local interactions [1]. Several exceptions to this argument have been recently found [2] in cellular automata (CA) models in which the average magnetization m (fraction of sites with state one) exhibits period three or near three with superimposed deterministic noise. In this paper we explain why the droplet growth argument given in [1] does not apply to the models in [2]. After summarizing the argument in [1] we describe the models in [2], and review known results. We continue with a report of our simulations and their implications. In particular, we observe a diffusionlike evolution of droplet-medium interfaces which provides a new mechanism for droplet breakdown and suggests that the droplet-growth equation proposed in [1] [Eq. (1) below] is inappropriate for the models in [2].

We begin by summarizing the argument in [1]. Consider a lattice system in which each node can be in one of three states, s_0, s_1, or s_2. There is a microscopic transition rule that results in asymptotic behavior (majority s_0) → (majority s_1) → (majority s_2) → (majority s_0) →⋯ This would yield spatially uniform states, but there is also some thermal noise which creates droplets of s_1 and s_2 in the configuration with majority s_0, and so on.

According to [1] the equation that describes the droplet size R, say of s_0 in s_1 is

$$\frac{dR}{dt} = -\frac{\sigma}{R} + h, \tag{1}$$

where σ is analogous to surface tension and the field h is positive if it favors s_0 and negative if it favors s_1. It is argued that generically $h \neq 0$. The exceptions are certain anisotropic rules, or the case of exactly two uniform phases. Therefore, the argument only applies to periods three or greater.

If $h > 0$ droplets of s_0 will grow in configurations with majority s_1; if $h < 0$ the opposite will happen. The same argument can be applied to the pairs s_0, s_2 and s_1, s_2. The conclusion is that droplet growth destroys spatial coherence, and the asymptotic states are an incoherent mixture of the three states. Simulations in [1] confirm this, showing such phenomena as spiral waves and phase dislocations. Period-three behavior was only observed in atypical (e.g., highly anisotropic) rules constructed to make the field term precisely zero.

The CA models studied systematically in [2] are dynamical systems in a three-dimensional (3D) to 5D (hyper)cubic lattice; sites can take values 0 or 1, and are updated synchronously. The evolution rule is *totalistic*, i.e., the next value depends on the sum of site values over a local neighborhood. In particular, sites are updated to be one if the sum of neighbors is in some intermediate range, or zero if the sum is too high or too low; specific examples are given below. While the models have no known physical meaning, their study can be profitable in the understanding of oscillating extended systems such as Rayleigh-Bénard convection and surface waves; they also have been associated with kinetic roughening and interface growth [3]. In many cases the correlations induced by determinism cause significant deviations from mean-field results, which predict a single-humped return map $m(t+1)$ vs $m(t)$, with m as defined above. For example, global oscillations in m with period three or near three, in apparent violation of the argument in [1], have been observed. The corresponding return maps, showing three dots with noise or a donut with noise, can be found in [2].

Several attempts [2–7] have been made to understand the phenomenology of these models. A second-order phenomenological equation [4] for m showed how irrational-period oscillations can arise in a discrete-time dynamical model and reproduced qualitatively the return map for m. Numerically measured site-site correlations [5] and cluster-expansion approximations [6] also reproduce qualitatively the magnetization return map. A noteworthy numerical study of noise and metastability in these systems is given in Ref. [7]. Presently the models with irrational frequencies are well understood: it has been suggested [3] that fluctuations in these systems are governed by the Kardar-Parisi-Zhang (KPZ) equation [8] for interface growth. This has been corroborated [9] by measuring spatial and temporal spin-spin correlations in these models, which show algebraic decay laws consistent with models that obey the KPZ equation. However, in the case of period-three oscillations much remains to be known; it has been suggested independently that different parts of these systems are uncoupled [10], or weakly coupled [11]; Ref. [12] showed strong evidence against the first of these two hypotheses.

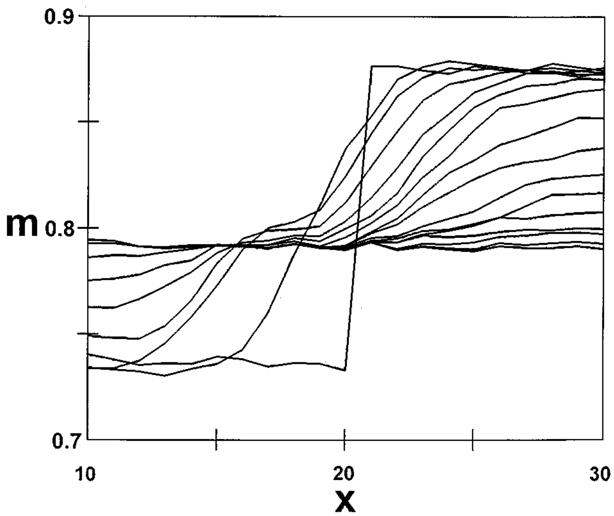

FIG. 1. Evolution of magnetization vs distance near a sharp interface between two inhomogeneous phases, averaged over 20 independent configurations. The four-dimensional model is described in the text. Density profiles are shown every 60 time steps.

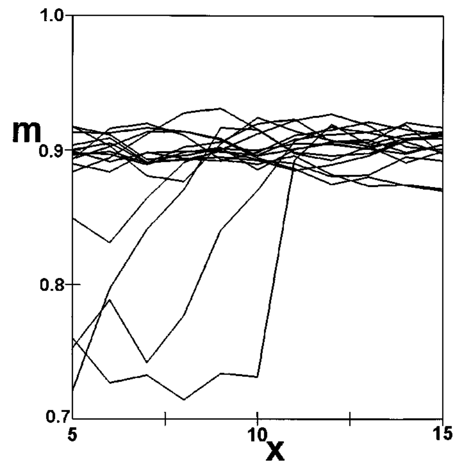

FIG. 2. Preliminary results of interface dynamics for the five-dimensional model described in the text. The axes are the same as for Fig. 1. Density profiles are shown every 60 time steps.

Two major differences appear between these and the models used in the argument in [1]: three phases with different magnetizations appear with only two states per site, which means that the phases are *inhomogeneous*, and there are no spontaneous thermal fluctuations, although the densities of the phases have small local fluctuations due to deterministic noise (see below). However, the rules are isotropic and period-three behavior appears in more than one specific model, which suggests that some step of the argument [1] fails. In what follows we study the validity of the droplet growth equation by creating a flat (infinite-radius) interface, which eliminates the first term in the right hand side of Eq. (1).

First we studied a four-dimensional model in which sites are updated to be 1 if the sum of the nine-site von Neumann neighborhood (center site and nearest neighbors) is between 3 and 8, and to be zero otherwise. In this model the magnetization oscillates between $m \sim 0.57$, 0.74, and 0.85 approximately. A randomly chosen initial condition with $m \sim 0.5$ was generated. Then we evolved the system for 1000 steps, to ensure that it had reached the period-three asymptotic state, and monitored it for a few more until we were sure it had reached a particular phase (say, $m \sim 0.57$). This was necessary because different initial conditions may reach the periodic state out of phase with each other. We saved that configuration and ran the system for one more time step. We then prepared a system with the left half in one phase and the right half in the other, thereby creating a flat interface between the two phases. We then allowed the system to evolve, and measured densities for the three-dimensional cuts parallel to the direction of the interface.

Figure 1 shows the time evolution of densities near the interface. Systems of size 40×20^3 were considered, with the interface chosen halfway along the longest dimension. Similar results [13] were obtained for each of the three possible interfaces, as well for a five-dimensional example described below. Densities are plotted every 60 time steps (20 complete cycles of the period-three oscillation), starting with a step function between $m \sim 0.74$ and $m \sim 0.85$. Only sites with values between 10 and 30 in the longest dimension are shown, i.e., those nearest the original interface.

The profiles are reminiscent of heat diffusion in an infinite region with the initial temperature distribution of a step function. However, a comparison of our simulations and the exact result (Chap. II in Ref. [14]) indicates that our model does not follow linear diffusion, even allowing for uncertainty in the simulations and drift. This drift, of about four lattice units to the left in 400 time steps, corresponds to a very small value of the field term, $h \sim 10^{-2}$, consistent with previous measurements in Ref. [7].

In order to check the generality of the diffusionlike phenomenon, we also did preliminary simulations of a 5D model which also exhibits period-three oscillations. In this case we have a totalistic rule over the 11-site von Neumann neighborhood. Sites are updated to one if the sum is between four and ten, and to zero otherwise. The results, shown in Fig. 2, confirm that this phenomenon indeed exists in the 5D model. In order to avoid problems specific to random number generators [15], initial conditions were generated independently with generators from three different families [16]. The graphs are mutually consistent (only one is shown), which gives us confidence in the results.

Based on these results of flat-interface simulation, the following picture emerges for the models of Ref. [2]: because of small fluctuations due to the inhomogeneity of the phases, small fluctuations in magnetization will develop at every time step. The gradients and droplet sizes [17] will not be very large. The droplet growth rate will be at most h, much less for small droplets because of the negative surface tension term. For a droplet of radius 3 with a 4% density difference [17], R will grow at most by one lattice unit in about 100 time steps. In that time, the density difference at the interface will drop to about 1% (see Fig. 1). Therefore, the diffusionlike mechanism near the interface tends to destroy droplets in time scales comparable to the growth term, preserving the stability of the inhomogeneous phase.

This picture has an additional mechanism not considered in the droplet growth Eq. (1), which explains why in the models under consideration the period-three oscillations are stable not only to intrinsic fluctuations but also to small amounts of added noise, as reported in [2,7]. Moreover, the results indicate that, for the models under consideration, Eq. (1) should be replaced by a partial differential equation for $m(x,t)$.

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[16] We used a combined linear congruential generator with long period and small distances between planes, a Tausworthe generator, and a well-known generalized feedback shift register used in several of the papers in Ref. [15].

[17] The probability of formation of droplets can be estimated by combinatorial sums. For example, if $m = 0.8$, the probability of a four-dimensional volume of radius 3 (400 sites) having magnetization that differs by more than 5% from m is less than 4%. Fluctuations of larger radius or magnetization difference occur with much smaller probabilities.